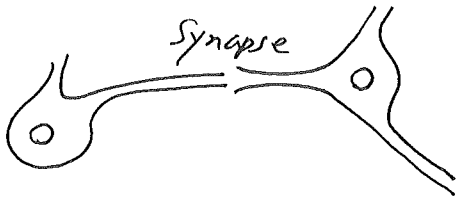
Hebbian Learning

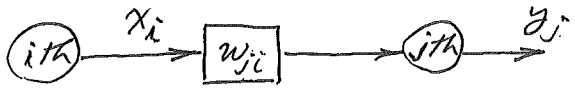
Donald O. Hebb, The Organization of Behavior, 1949

Hebb's Postulate:

"When an axon of cell A is near enough to excite a cell B and repeatedly or persistantly takes part in firing it, some growth process or metabolic changes takes place in one or both cells such that A's efficiency, as one of the cells firing B, is increased."



Neuron A Neuron B



Hence Hebb's principle will increase the common weight w_{ji} when there is activity flowing from the ith neuron to the jth neuron:

$$\Delta w_{ji} = \eta\, x_i\, y_j \quad : \text{Hebb's rule}$$

Note that, unlike the BPA, there is no desired output required in Hebbian learning. To apply Hebb's rule, only the input signal needs to flow through the NN: — unsupervised learning.

Thus Hebbian learning updates the weights according to

$$w(n+1) = w(n) + \eta\, x(n)\, y(n)$$

where n is the iteration number and η is a step size. For a linear processing element, $y = wx$, so

$$w(n+1) = w(n)\,[1 + \eta x^2(n)]$$

Weight will increase with the number of iterations without bounds. Hence Hebbian learning is intrinsically unstable. In biology this is not a problem since there are natural nonlinearities that limit the synaptic efficacy (chemical depletion, dynamic range, etc.).

Multiple-input PE:

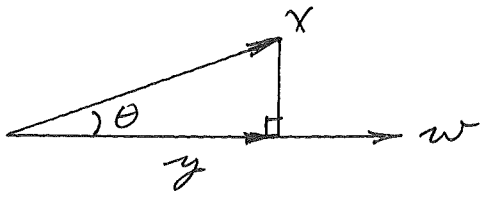
$$y = \sum_{i=1}^{n} w_i x_i$$

Hebb's rule, implies

$$\Delta w = \eta \begin{bmatrix} x_1 y \\ \vdots \\ x_N y \end{bmatrix}$$

Note the output is, in vector notation,

$$y = w^T x = x^T w = \langle x, w \rangle : \text{inner product}$$



For normalized inputs and weights, a large y means input x is "close" to the direction of the weight vector, i.e., x is in the neighborhood of w.

Associative Memory: Linear associater is

$$y_j = \sum_{i=1}^{n} w_{ji} x_i$$

or in the vector notation,

$$y = W x.$$

The task of an associative memory is to learn P pairs of prototype input/output vectors:

$$\{x_1, t_1\}, \{x_2, t_2\}, \cdots, \{x_P, t_P\}.$$

Hebb's rule is

$$w_{ji}^{new} = w_{ji}^{old} + \eta\, y_{jp}\, x_{ip}$$

for the pth pattern. This is an unsupervised learning rule. For the supervised Hebb's rule, we substitute the target with the actual output:

$$w_{ji}^{new} = w_{ji}^{old} + \eta\, t_{jp}\, x_{ip}$$

In vector form, with $\eta = 1$ for simplicity,

$$W^{new} = W^{old} + t_p X_p^T$$

Assume $W(0) = 0$, initialal weight, apply each pattern once, then

$$W = t_1 X_1^T + t_2 X_2^T + \cdots + t_p X_p^T = \sum_{p=1}^{P} t_p X_p^T$$

In matrix form,

$$W = [t_1 \; t_2 \cdots t_p] \begin{bmatrix} X_1^T \\ X_2^T \\ \vdots \\ X_p^T \end{bmatrix} = T X^T$$

where

$$T = [t_1 \; t_2 \cdots t_p] \quad X = [x_1 \; x_2 \cdots x_p]$$

<u>Performance</u>: Assume x_p vectors are orthonormal.

Then

$$y = W x_k = \left(\sum_{p=1}^{P} t_p x_p^T \right) x_k = \sum_{p=1}^{P} t_p (x_p^T x_k)$$

Since x_p are orthonormal,

$$(x_p^T x_k) = 1 \qquad p = k$$
$$= 0 \qquad p \neq k$$

Therefore,

$$y = W x_k = t_k$$

$\Rightarrow$ The output of NN is equal to the target.

If not orthogonal input vectors, then we have error. Assume x_p vector is unit length, but not orthogonal. Then

$$y = W x_k = t_k + \underbrace{\sum_{p \neq k} t_p (x_p^T x_k)}$$

error : depends on correlation between input patterns

Example: Input patterns are

$$x_1 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}, \; t_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad x_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}, \; t_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Check that the two input patterns are orthonormal.

The weight matrix would be

$$W = T X^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Test:

$$W x_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = t_1$$

$$W x_2 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = t_2$$

Example: Apple & orange recognition.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ (orange)}, \qquad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ (apple)}$$

Note these are not orthogonal.

Normalize these inputs and choose desired outputs -1 and 1,

$$x_1 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}, \; t_1 = -1; \quad x_2 = \begin{bmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix}, \; t_2 = 1$$

The weight matrix becomes

$$W = T X^T = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.5774 & -0.5774, & -0.5774 \\ 0.5774 & 0.5774 & -0.5774 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1.547 & 0 \end{bmatrix}$$

Test:

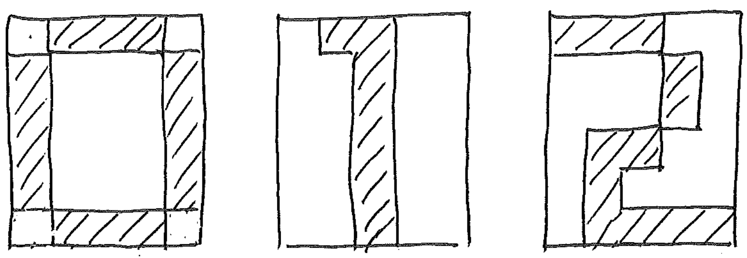
$$Wx_1 = [0 \;\; 1.547 \;\; 0]\begin{bmatrix}0.5774\\-0.5774\\-0.5774\end{bmatrix} = [-0.8932] \sim t_1$$

$$Wx_2 = [0 \;\; 1.547 \;\; 0]\begin{bmatrix}0.5774\\0.5774\\-0.5774\end{bmatrix} = [0.8932] \sim t_2$$

The outputs are close, but do not quite match the targets.

Application (Pattern Recognition):

Autoassociative Memory: Desired output vector is equal to the input vector (i.e., $t_p = x_p$).



white : -1
dark : 1

Scan each 6×5 grid one column at a time.

x_1, t_1 $\qquad$ x_2, t_2 $\qquad$ x_3, t_3

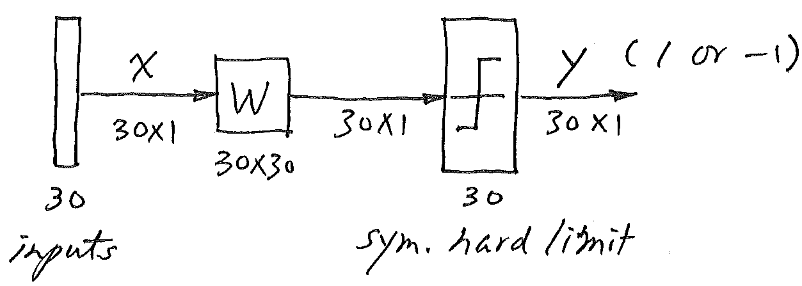
$x_1 = [-1\ 1\ 1\ 1\ 1\ -1 \,\vdots\, 1\ -1\ -1\ -1\ -1\ 1 \,\vdots\, 1\ -1\ \cdots\ 1\ -1]^T$, digit "0"

$x_2 = [\qquad]^T$, digit "1"

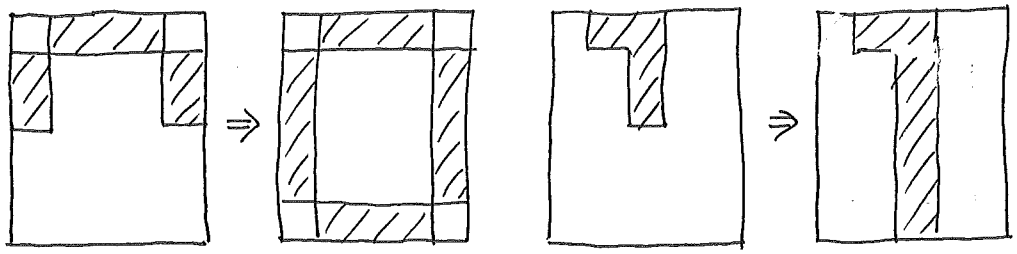
$x_3 = [\qquad]$, digit "2"

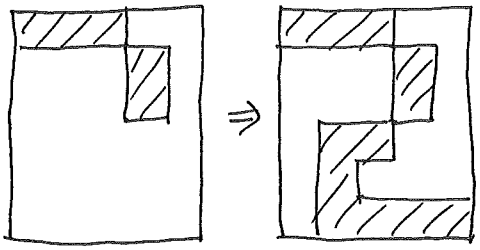
Hebb's rule $\Rightarrow$

$$W = x_1 x_1^T + x_2 x_2^T + x_3 x_3^T \qquad (x_p \text{ replaces } t_p)$$



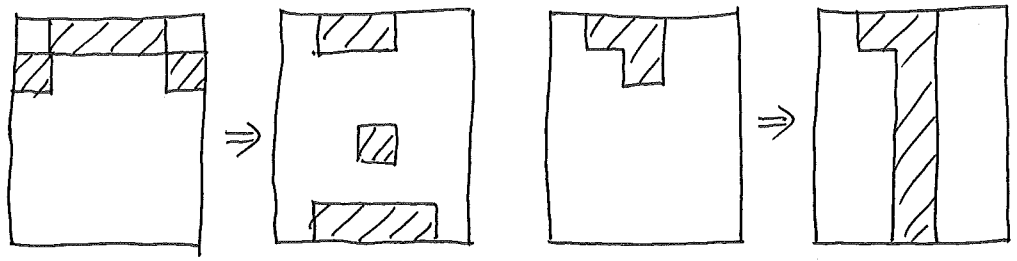
Recall patterns, even when corrupted patterns are provided as inputs.

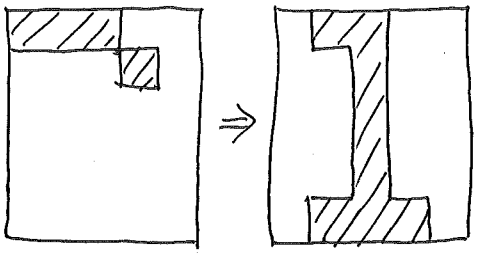




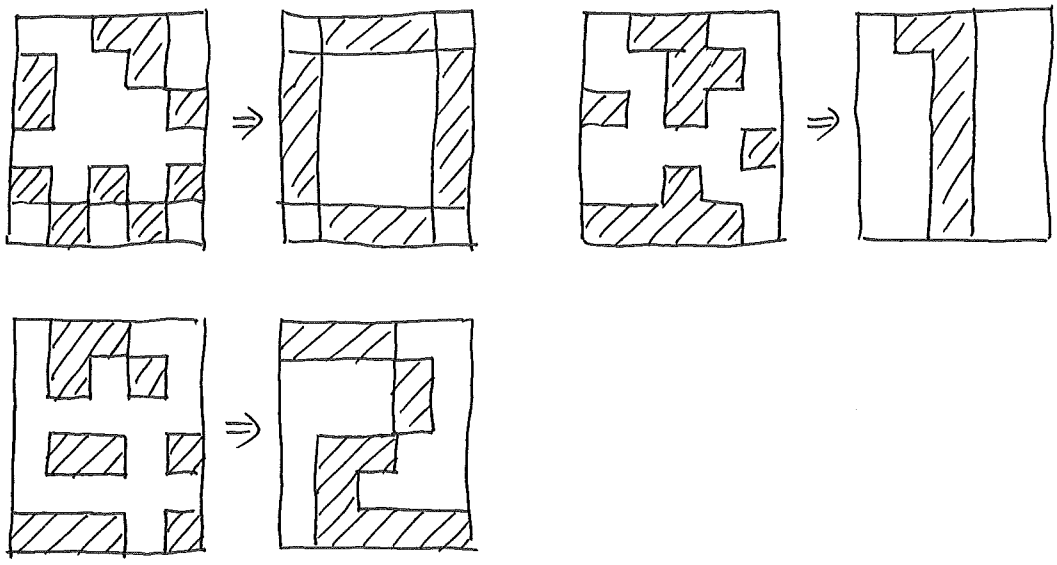
recovery for inputs with lower half occluded. (50%)

For lower 2/3 occluded (67%), error occurs!





For randomly changed (7 elements) pattern:



Variations of Hebbian learning:

Basic rule:

$$W^{new} = W^{old} + t_p x_p^T$$

If there are too many patterns in the training set, the weight matrices will have large elements.
Use the learning rate to limit the amount of increase:

$$W^{new} = W^{old} + \eta t_p x_p^T$$

Add a decay term, so that the learning rule behaves like a smoothing filter, remembering the most recent inputs more clearly:

$$W^{new} = W^{old} + \eta t_p x_p^T - \gamma W^{old} = (1-\gamma) W^{old} + \eta t_p x_p^T$$

as $\gamma \to 0$, it becomes the standard rule,
as $\gamma \to 1$, the learning law quickly forget old inputs, remembering the most recent inputs. This keeps the weight matrix from growing without bound.

If the desired output is replaced by the difference between desired and actual output,

$$w^{new} = w^{old} + \eta (t_p - y_p) x_p^T$$

: Delta rule (Widrow-Hoff algorithm)

Replace the desired output with the actual output:

$$w^{new} = w^{old} + \eta\, y_p\, x_p^T$$

: unsupervised learning.